ν = the constant in Equation (1), $(= k/\rho)$
σ = the electrical conductivity of the fluid
ξ, η = the Crocco's variables defined by Equations (4)
τ = the dependent variable defined by Equation (7)
λ = independent variable defined by Equation (8)
ζ = independent variable defined by Equation (15)
τ_0 = the shear stress at the wall

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Spatial Formulation of the Orr-Sommerfeld Equation for Thin Liquid Films Flowing Down a Plane

The linear stability problem for thin liquid films flowing down a plane is formulated in terms of disturbances which are harmonic in space and time and grow or decay spatially. A closed-form analytical solution to the resulting spatial formulation of the Orr-Sommerfeld equation is shown to compare more favorably with wave property data than do solutions to the classical temporal formulation of this stability problem. The predictions of Gaster that the spatial and temporal formulations will be equivalent only for weakly amplified nondispersive disturbances are confirmed for falling film flow.

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SCOPE

The flow of a thin liquid film down a plane is frequently used as an idealized hydrodynamic description of the complex flow occurring in many types of engineering equipment and processes; as such, it can be used to estimate the liquid-side heat and/or mass transfer coefficients and to evaluate other performance characteristics of the equipment or process. The tendency of liquid film flow to be unstable to small disturbances is initially manifest as two-dimensional traveling waves which can greatly influence the performance of any equipment involving unstable film flow. Our ability to design such equipment will be improved if we can predict the occurrence and

wave properties of unstable film flow.

The occurrence and properties of two-dimensional waves on thin liquid film flow are predicted in the studies of Benjamin (1957), Yih (1963), and Anshus and Goren (1966). However, the results of these studies were not in quantitative agreement with the precise wave property data of Krantz and Goren (1971). This lack of quantitative agreement may arise from the fact that all of the aforementioned studies assume that the waves arise from disturbances which are harmonic in space and time, and grow or decay temporally, whereas in fact the disturbances grow or decay spatially.

The objectives of this study were to determine the stability characteristics of thin film flow assuming that the waves arise from disturbances which are harmonic in space and time, and grow or decay spatially, and to ascertain whether the results of these analyses represented a significant improvement over prior analysis of this flow which assumed that the disturbances grow or decay temporally.

CONCLUSIONS AND SIGNIFICANCE

The instability of thin liquid films with respect to small disturbances which are harmonic in space and time, and grow or decay spatially was analyzed. The predictions were compared to data on the growth and decay rates and wave velocities of two-dimensional traveling waves on thin liquid films of two White Oils.

The predictions of the theory developed here for spatially growing or decaying disturbances agreed well with all the available wave property data. This represents a significant improvement in our ability to understand and predict the wave properties of small disturbances on liquid films, as prior theories could not predict wave growth and decay quantitatively.

A detailed comparison between the predictions of the spatial and temporal formulations of the stability problem revealed that solutions to the temporal formulation could yield errors as large as 45% in the growth or decay rates. The solutions to the two formulations were found to be equivalent only when the waves grow or decay very slowly and have a phase velocity which is not a strong function of the wave number.

The results obtained here suggest that the predictions of many solutions to the temporal formulation of stability problems developed in the literature should be used with considerable caution. Nearly all flow instabilities involve spatially growing disturbances; however, most of these flows have been analyzed assuming the disturbances to grow temporally. In the case of thin liquid film flow the greatest difference between the predictions of the temporal formulation and the correct spatial formulation of the problem occurs for the most highly amplified wave properties. This is particularly significant since the most highly amplified wave is the one which should be observed in an unstable flow in the absence of any imposed disturbances of controlled frequency. Our results suggest that many of the solutions to the temporal formulation of stability problems presented in the literature should be redeveloped in terms of the correct spatial formulation.

Thin liquid films flowing down a plane have been shown to be unstable with respect to infinitesimal amplitude, two-dimensional disturbances by Benjamin (1957) and Yih (1963). These investigators developed exact solutions to the linearized Orr-Sommerfeld equation, which are restricted to small Reynolds numbers. Approximate analytical solutions not restricted to small Reynolds numbers have been developed by Anshus and Goren (1966) and Krantz and Goren (1971). Numerical solutions have been developed by Whitaker (1964) and by Sternling and Towell (1965). No attempt will be made here to review in detail these and other solutions to the linear stability problem for thin liquid films; a recent review is given by Krantz and Goren (1971).

This paper is concerned with the anomaly that none of the aforementioned solutions can quantitatively predict the properties of small amplitude two-dimensional waves on thin liquid film flow. This lack of agreement between theory and experiment has been rather casually attributed to the fact that linear stability theory is restricted to waves of infinitesimal amplitude, whereas any observable waves must necessarily have a finite amplitude. It will be shown here that the lack of agreement between theory and experiment results from the fact that all solutions for falling film flow (with the exception of the very approximate solution of Krantz and Goren) assume that the disturbances or waves grow or decay temporally, when in fact they grow or decay spatially.

Figure 1 shows a two-dimensional traveling wave of wave length λ_r, wave velocity c_r, and instantaneous amplitude $\eta(x, t)$ imposed on a falling film of mean thickness h_0 flowing down a plane inclined at an angle β to the horizontal due to gravitational acceleration g. Note that if this wave is unstable, its amplitude increases spatially not temporally; that is, at an instant in time the wave increases in amplitude in the streamwise coordinate. At a fixed point in space the amplitude of the wave oscillates harmonically; however, the maximum peak-to-peak amplitude does not change in time. Thus it is clear that travel-

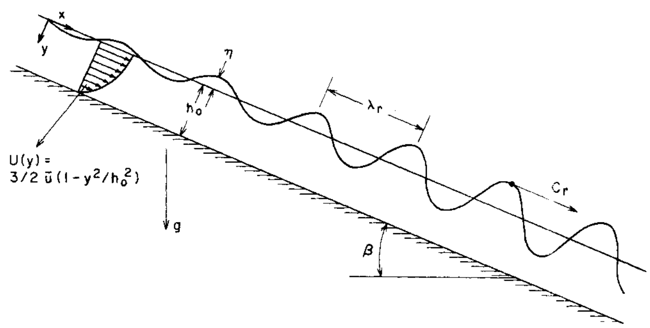

Fig. 1. An unstable, two-dimensional traveling wave on thin liquid film flow.

ing waves on thin liquid film flow grow or decay spatially, not temporally as has been assumed by most investigators.

At this point it is instructive to consider the formal difference between what we will henceforth refer to as the temporal and spatial formulations of the linearized Orr-Sommerfeld equation. Infinitesimal two-dimensional waves which are harmonic in space and time and grow or decay temporally are described by a disturbance stream function of the form

$$\psi(x, y, t) = \phi(y) \exp [i\alpha_r(x - c_r t)] \exp (\alpha_r c_i t) \quad (1)$$

or equivalently

$$\psi(x, y, t) = \phi(y) \exp [i\alpha_r(x - ct)] \quad (2)$$

where $\phi(y)$ is the dimensionless amplitude of the stream function; α_r is the dimensionless real wave number; $c = c_r + ic_i$ is the dimensionless complex wave velocity; c_r is the real wave velocity; and, c_i is related to the temporal amplification factor $\alpha_r c_i$. All quantities are nondimensionalized with respect to the film thickness h_0 and average velocity $\overline{U}$ of the undisturbed laminar flow.

Infinitesimal two-dimensional waves which are harmonic in space and time, and grow or decay spatially are described by a disturbance stream function of the form

$$\psi(x, y, t) = \phi(y) \exp [i(\alpha_r x - \omega_r t)] \exp (-\alpha_i x) \quad (3)$$

or equivalently

$$\psi(x, y, t) = \phi(y) \exp [i(\alpha x - \omega_r t)] \quad (4)$$

where $\alpha = \alpha_r + i\alpha_i$ is the dimensionless complex wave number; α_i is the spatial amplification factor; and ω_r is the dimensionless real angular frequency.

For quite some time investigators thought that predictions of the temporal formulation could be applied to spatially growing or decaying waves based on a transformation suggested by Schubauer and Skramstad (1949) given by $t = x/c_r$. This transformation implies that the spatial amplification factor equivalent to the temporal amplification factor is given by

$$-\alpha_i = \alpha_r c_i / c_r \quad (5)$$

However, Gaster (1965) has shown that this transformation can be justified only for weakly amplified nondispersive disturbances, that is, disturbances for which $|c_i| << c_r$, $|\alpha_i| << \alpha_r$ and whose phase velocity is not a function of wave number, which implies $dc_r/d\alpha_r = 0$. Waves on falling films may be highly amplified and have phase velocities which are strongly dependent on wave number. Thus it would appear that the lack of agreement between predicted and measured wave properties may be due to the fact that the predictions are those of solutions to the temporal formulation of the Orr-Sommerfeld equation. In the following sections the spatial formulation of the linear stability problem for falling film flow and a closed-form analytical solution are developed.

SPATIAL FORMULATION OF THE LINEAR STABILITY PROBLEM

If the disturbance stream function given by Equation (4) is substituted into the linearized Navier-Stokes equations and the pressure is cross-differentiated and eliminated between the x- and y-momentum equations, the spatial formulation of the Orr-Sommerfeld equation is obtained:

$$\phi'''' - 2\alpha^2\phi'' + \alpha^4\phi = i\alpha N_{Re}[(U - \omega_r/\alpha)(\phi'' - \alpha^2\phi) - U''\phi] \quad (6)$$

where $U = 3/2 \cdot (1 - y^2)$ and the primes denote differentiation with respect to y. The linearized boundary conditions and kinematic surface condition appropriate to wavy film flow are given by

$$\phi = 0 \quad \text{at} \quad y = 1 \quad (7)$$

$$\phi' = 0 \quad \text{at} \quad y = 1 \quad (8)$$

$$\phi'' + [\alpha^2 - 3/\omega_r/\alpha - 3/2)]\phi = 0 \quad \text{at} \quad y = 0 \quad (9)$$

$$[\alpha(3\cot\beta + \alpha^2 N_{We} N_{Re})/(\omega_r/\alpha - 3/2)]\phi - \alpha[N_{Re}(\omega_r/\alpha - 3/2) + 3\alpha i]\phi' + i\phi''' = 0 \quad \text{at} \quad y = 0 \quad (10)$$

$$-\phi(0) = (3/2 - \omega_r/\alpha)A \quad (11)$$

where A is the dimensionless wave amplitude one-half peak-to-peak. All quantities in the above equations and in those appearing elsewhere are nondimensionalized with respect to the film thickness and average velocity of the undisturbed flow. This nondimensionalization introduces the Reynolds number $N_{Re} = \overline{U}h_0/\nu$ and surface tension group $N_\zeta = (3/g\nu^4)^{1/3}/\rho$. Equations (7) and (8) express the no-flow and no-slip conditions at the solid boundary. Equation (9) expresses that in the absence of any influence of the gas phase, there can be no shear stress at the free surface. Equation (10) balances the normal stress at the wavy free surface with the pressure and surface tension terms.

SOLUTIONS TO THE SPATIAL FORMULATION OF THE ORR-SOMMERFELD EQUATION

Equations (6) to (11) corresponding to the spatial formulation of the Orr-Sommerfeld equation differ from the classical temporal formulation only in that α replaces α_r, and ω_r/α replaces c wherever they appear in the temporal formulation. Since transforming to the spatial formulation only alters the parameters of the Orr-Sommerfeld equation and its associated boundary conditions, any solutions developed for the classical temporal formulation can be applied to the spatial formulation by substituting $\alpha = \alpha_r$ and $\omega_r/\alpha = c$ in the resulting solution for ϕ. The parametric substitution involved in transforming from the temporal to the spatial formulation, however, does influence the solution of the eigenfunction. The latter results from substituting the solution of the Orr-Sommerfeld equation into the boundary conditions and thus constitutes a relationship between the parameters of the problem. In the temporal formulation the complex wave velocity is the eigenvalue and the real wave number is a parameter; in the spatial formulation the complex wave number is the eigenvalue and the real wave frequency is a parameter. The solutions to these two eigenvalue problems will of course differ.

The spatial analogues of the temporal solutions of Benjamin (1957) and Yih (1963) have been developed in the thesis of Owens (1972); however, they will not be considered here since they are restricted to small wave numbers and Reynolds numbers. These conditions are not satisfied in most of the experiments for which detailed wave property data are available.

The only solutions to the spatial formulation of the linear stability problem for thin liquid films have been developed by Krantz and Goren who employed two different integral methods. Details of these solutions are available elsewhere in the literature (Krantz and Goren, 1971).

Anshus and Goren (1966) developed an exact analytical solution to an approximate form of the temporal formulation of the Orr-Sommerfeld equation. Anshus and Goren reasoned that the interfacial region controlled the stability of film flow. On this basis they made the substitution $U = 3/2$ in the Orr-Sommerfeld equation, thus reducing it to an ordinary differential equation with constant rather than variable coefficients, which permitted them to obtain a closed-form analytical solution. Comparison of this solution with Sternling and Towell's numerical solution to the temporal formulation of the exact Orr-Sommerfeld equation showed the surface approximation to be quite accurate for all practical values of the flow parameters. Hopefully the surface approximation can be used to obtain a closed-form analytical solution of the spatial formulation of the Orr-Sommerfeld equation which will agree equally well with the numerical solution to the spatial formulation. Note however that a numerical solution to the spatial formulation has not been developed as yet.

The eigenfunction corresponding to the solution to the spatial formulation of the Orr-Sommerfeld equation employing the surface approximation is given by the determinant

$$\begin{vmatrix} \sin\beta_1 & \cos\beta_1 & \sin\beta_2 & \cos\beta_2 \\ \beta_1\cos\beta_1 & -\beta_1\sin\beta_1 & \beta_2\cos\beta_2 & -\beta_2\sin\beta_2 \\ 0 & (3/2-\omega_r/\alpha)(\alpha^2-\beta_1^2)+3 & 0 & (3/2-\omega_r/\alpha)(\alpha^2-\beta_2^2)+3 \\ \begin{matrix}(3/2-\omega_r/\alpha)\beta_1\cdot \\ [i\alpha N_{Re}(3/2-\omega_r/\alpha) \\ +\beta_1^2+3\alpha^2]\end{matrix} & i\alpha^3 N_{We} N_{Re} & \begin{matrix}(3/2-\omega_r/\alpha)\beta_2\cdot \\ [i\alpha N_{Re}(3/2-\omega_r/\alpha) \\ +\beta_2^2+3\alpha^2]\end{matrix} & i\alpha^3 N_{We} N_{Re} \end{vmatrix} = 0 \qquad (12)$$

where

$$\beta_1 = \left\{ -\alpha^2 - \frac{1}{2} i\alpha N_{Re}\,(3/2 - \omega_r/\alpha) + \left[3i\alpha N_{Re} + \left(\frac{1}{2} i\alpha N_{Re}(3/2 - \omega_r/\alpha) \right)^2 \right]^{1/2} \right\}^{1/2}$$

$$\beta_2 = \left\{ -\alpha^2 - \frac{1}{2} i\alpha N_{Re}\,(3/2 - \omega_r/\alpha) - \left[3i\alpha N_{Re} + \left(\frac{1}{2} i\alpha N_{Re}(3/2 - \omega_r/\alpha) \right)^2 \right]^{1/2} \right\}^{1/2}$$

This eigenfunction appropriate to the spatial formulation can be obtained from the eigenfunction of Anshus and Goren for the temporal formulation by making the substitutions $\alpha = \alpha_r$ and $\omega_r/\alpha = c$. Note, however, that the eigenvalues of Equation (12) correspond to the complex wave numbers whereas the eigenvalues of the temporal growth eigenfunction correspond to the complex wave velocities. A complex Newton-Raphson technique was used to solve Equation (12) for the complex eigenvalue α as a function of the parameters of the system.

COMPARISON WITH WAVE PROPERTY DATA

In order to ascertain whether solutions to the spatial formulation of the Orr-Sommerfeld equation are significantly better than solutions to the temporal formulation, it is necessary to have precise data on the amplification rates and phase velocities of both stable and unstable waves. Such data have been presented recently by Krantz and Goren (1971) who measured the wave properties of two-dimensional disturbances of controlled amplitude and frequency. In order to compare their data with the various solutions to the temporal growth formulation of the linear stability problem, Krantz and Goren transformed their measured spatial amplification factors to equivalent temporal amplification factors using the transformation suggested by Schubauer and Skramstad given by Equation (5). Hence the equivalent temporal amplification factor was obtained from the measured wave velocity c_r and spatial amplification factor α_i. This choice for presenting the data is unfortunate for two reasons: Krantz and Goren indicate that their measurements of α_i were far more precise than those of c_r, thus the error in c_r is introduced into the reported values of $\alpha_r c_i$; in addition, the transformation given by Equation (5) has been shown to be valid only for weakly amplified nondispersive waves, that is, for waves whose phase velocity is independent of wave number. Neither of these conditions are satisfied by most of Krantz and Goren's data. They found that the best linear stability theory solutions underpredicted the temporal amplification factors by 15 to 45% and underpredicted the wave velocity by as much as 5%. Typical results for the temporal amplification factor $\alpha_r c_i$ and wave velocity c_r as a function of real wave number are shown in Figure 2. This figure shows the predictions of the analytical solutions of Benjamin (1957), Yih (1963), Krantz and Goren (1971), and Anshus and Goren (1966). It is included here for later comparison with a similar plot showing the spatial amplification factor and corresponding solutions to the spatial formulation of the Orr-Sommerfeld equation.

Krantz and Goren concluded that the available solutions to the linear stability problem are qualitatively correct but are unable to predict the wave properties quantitatively. They attributed this lack of agreement between theory and data to the approximations involved in the various solutions to the temporal formulation of the Orr-Sommerfeld equation. In the following section we show that the analytical solution to the spatial formulation of the approximate Orr-Sommerfeld equation given by Equation (12) is capable of predicting the wave properties of small amplitude two-dimensional waves quantitatively.

In Figures 3 and 4 the values of the spatial amplification factor, $-\alpha_i$, and wave velocity ω_r/α_r obtained from the data reported by Krantz (1968) are compared directly with two solutions to the spatial growth formulation of the linear stability problem. Figure 3 is for Chevron No. 15 U.S.P. White Oil ($\mu = 1.46$ poise; $\rho = 0.868$ g/cc; $\sigma = 30.8$ dynes/cm; at 25°C) and represents the same raw data as were used in Figure 2; Figure 4 is for Chevron No. 5 N.F. White Oil ($\mu = 0.499$ poise; $\rho = 0.853$ g/cc; $\sigma = 29.5$ dynes/cm; at 25°C). The curve denoted by O-K corresponds to the solution to the spatial formulation of the approximate Orr-Sommerfeld given by Equation (12); that denoted by K-G corresponds to the integral-eigenvalue solution to the spatial formulation of the Orr-Sommerfeld equation developed by Krantz and Goren.

Figures 3 and 4, which are representative of all the data of Krantz and Goren, indicate that the solution to the spatial formulation of the Orr-Sommerfeld equation given by Equation (12) predicts the spatial amplification factor quantitatively. A comparison between Figures 2 and 3, which represent the same raw data, clearly establishes the superiority of the spatial growth formulation.

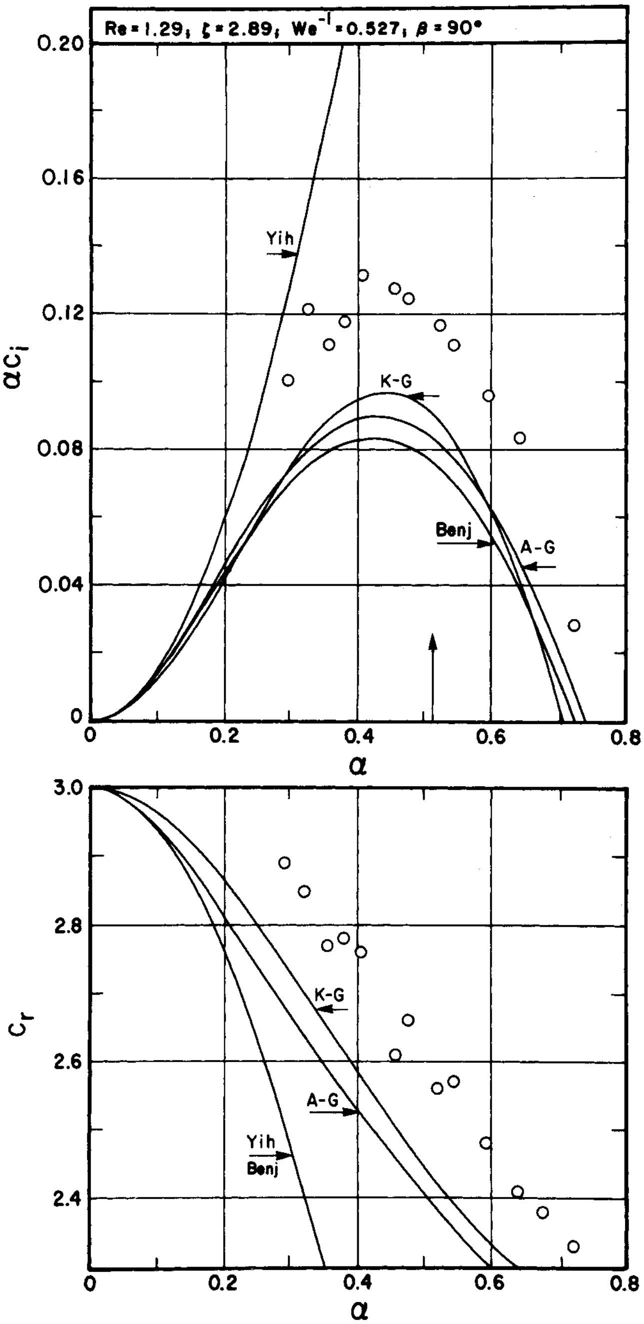

Fig. 2. Temporal amplification factor and wave velocity as functions of wave number for $N_{Re} = 1.29$, $N_\zeta = 2.89$, $N_{We}{}^{-1} = 0.527$, $\beta = 90°$.

Recall here that the solution denoted by A-G in Figure 2 and that denoted by O-K in Figure 3 differ only in that the former is a solution to the temporal formulation, whereas the latter is a solution to the spatial formulation; they both employ the same method of solution, namely, that proposed by Anshus and Goren.

The solution denoted by K-G in Figures 3 and 4 is at best qualitatively correct. It would appear that the fourth-degree polynomial assumed for the stream function in this integral solution is inexact for these flow conditions.

The solutions corresponding to the spatial formulation appear to do no better in predicting the wave velocity than do the temporal solutions, as both underpredict by approximately 5%. This is believed to be a result of an increased wave velocity of the finite-amplitude waves associated with these data. The nonlinear stability theories of Nakaya and Takaki (1967), Lin (1969), and Gjevik (1970) predict that the wave velocity of the infinitesimal waves will increase as the waves reach finite amplitude.

In Figures 2 to 5 the vertical arrow indicates the most highly amplified wave found experimentally by allowing room disturbances to grow in the absence of any controlled disturbance. Note that the vertical arrow corresponds closely with the maximum in the spatial amplification curve predicted by the solution to the spatial formulation of the Orr-Sommerfeld equation given by Equation (12). The most highly amplified wave is more closely predicted by the spatial growth solution than by the temporal growth solution of the Orr-Sommerfeld equation which underpredicts the wave number at which these waves will occur, as can be seen by comparing Figures 2 and 3. In the temporal formulation the most highly amplified wave corresponds to a maximum in $\alpha_r c_i$ whereas both experimentally and in the spatial formulation these waves correspond to a maximum in $-\alpha_i$. The temporal amplification factor is approximately equal to $-c_r\alpha_i$; since in the region of the most highly amplified wave the wave velocity decreases as the wave number increases, the maximum in the temporal amplification curve will occur at a lower wave number than will the maximum in the spatial amplification curve as is observed in Figure 2. Owens (1972) has compared the predictions of Equation (12) for the most highly amplified wave number with all the White

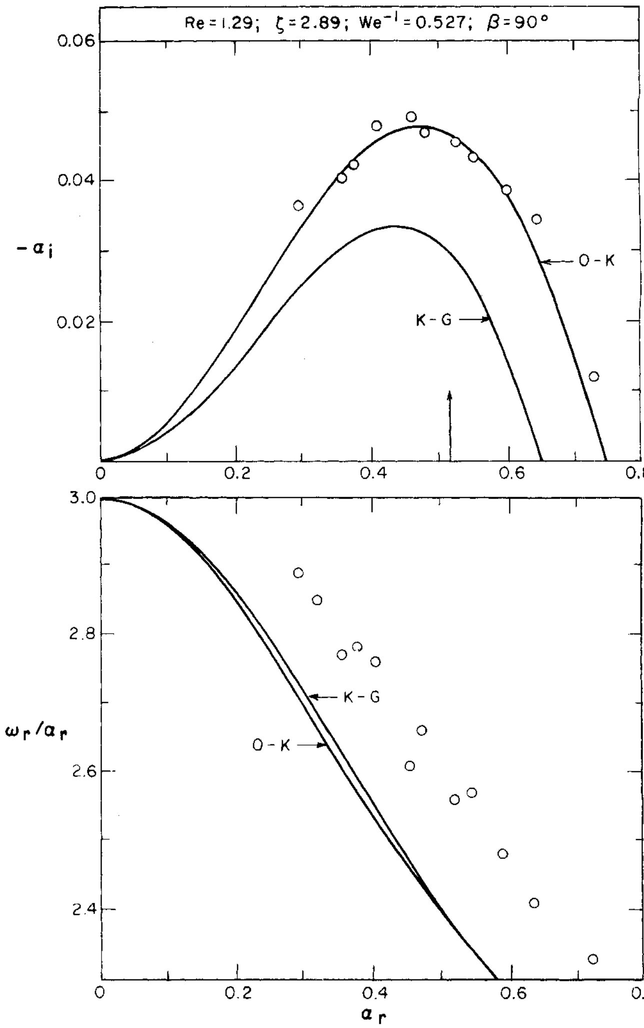

Fig. 3. Spatial amplification factor and wave velocity as functions of wave number for $N_{Re} = 1.29$, $N_\zeta = 2.89$, $N_{We}{}^{-1} = 0.527$, $\beta = 90°$.

Oil data of Krantz and Goren and all the available data for water. Equation (12) predicts the White Oil data quantitatively; however, the data for water scatter considerably thus making it difficult to conclude whether Equation (12) does better in predicting these data than does the temporal growth solution of Anshus and Goren.

Figures 2 and 3 provided a comparison between two identical methods of solving the Orr-Sommerfeld equation which differ only in that one solves the spatial formulation whereas the other solves the temporal formulation. However, this does not provide a direct comparison between solutions to the two formulations since Figure 2 shows the temporal amplification factor whereas Figure 3 shows the spatial amplification factor. It is of interest to compare the values of the spatial amplification factor predicted by the temporal formulation with those of the spatial formulation. Spatial amplification factors can be obtained from the temporal amplification factors shown in Figure 2 by applying the approximate transformation given by Equation (5), that is, $-\alpha_i = \alpha_r c_i / c_r$ where $\alpha_r c_i$ and c_r are the temporal amplification factor and wave velocity predictions of the temporal formulation. This comparison is made for the flow conditions of Figures 2 and 3 and is shown in Figure 5. The curve denoted by A-G is the temporal solution of Anshus and Goren and that denoted by O-K is the spatial solution given by Equation (12). This figure clearly indicates that the solution to the temporal formulation will not give quantitative predictions of the amplification factor for spatially growing or decaying waves under all conditions. Note that the two solutions agree for very small wave numbers and for the neutrally stable wave, in agreement with the predictions of Gaster (1965). At small wave numbers the waves are weakly amplified, that is, $|\alpha_i| << \alpha_r$, and the phase velocity is nearly constant, thus $dc_r/d\alpha_r \simeq 0$ corresponding to nondispersive waves. At neutrally stable wave number there is no distinction between the spatial and temporal formulations of the linear stability problem since this wave neither grows nor decays.

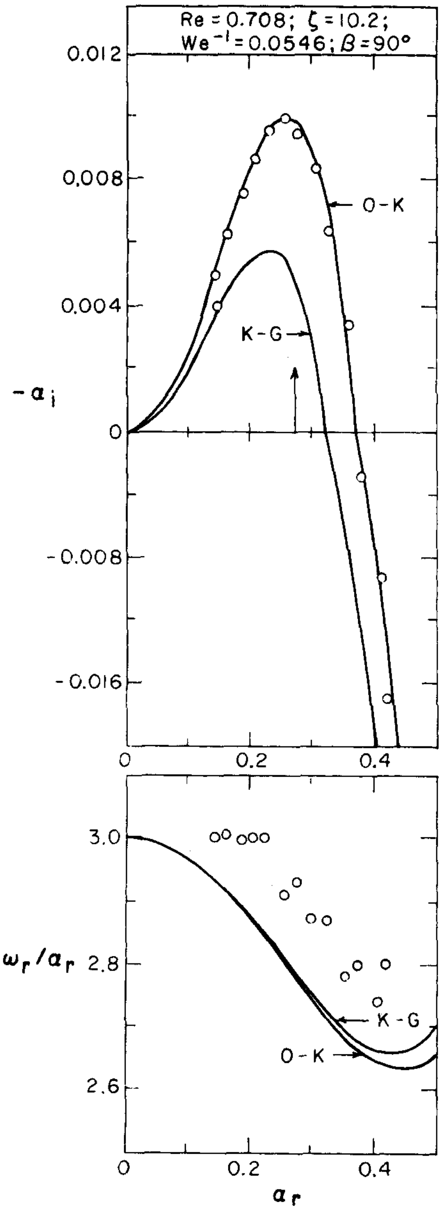

Fig. 4. Spatial amplification factor and wave velocity as functions of wave number for $N_{Re} = 0.708$, $N_\zeta = 10.2$, $N_{We}^{-1} = 0.0546$, $\beta = 90°$.

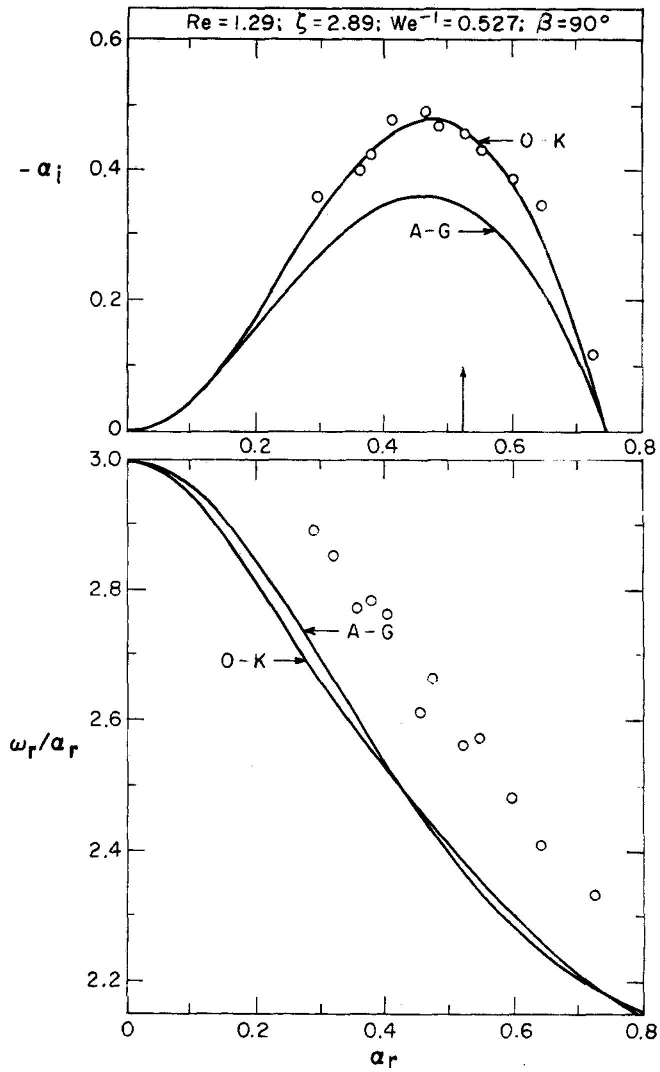

Fig. 5. A comparison of solutions to the temporal and spatial formulations of the linear stability problem for falling films for $N_{Re} = 1.29$, $N_\zeta = 2.89$, $N_{We}^{-1} = 0.527$, $\beta = 90°$.

DISCUSSION

The results presented here confirm for falling film flow the predictions of Gaster for the range of validity of the transformation suggested by Schubauer and Skramstad for comparing the predictions of temporal growth theories with data on spatially growing waves. There are many unstable flows which involve spatially growing disturbances such as jets, boundary layers, open-channel flows, flame sheets, vortex streets, flows involving Kelvin-Helmholtz instabilities, as well as other flows, which have been analyzed using the classical temporal formulation of the linear stability problem. In many cases one is only interested in the minimum critical Reynolds number for the growth of two-dimensional waves. This is readily obtained from the neutral stability curve which is correctly described by solutions to the temporal formulation of the linear stability problem since temporal theory is equivalent to spatial theory for neutrally stable waves. However, in some flows one is interested in the wave properties of the unstable waves. For example, in the case of jets one may wish to predict how rapidly the jet will break up and the characteristic size of the droplets formed. In the case of Kelvin-Hemholtz instabilities one may be interested in predicting the spatial growth of the wave trails generated by a time-periodic disturbance at a point in space. In such cases the predictions of the temporal formulation of the linear stability problem should be interpreted with caution since the unstable flow is properly described by the spatial formulation. In all cases it is no more difficult to solve the spatial formulation of the Orr-Sommerfeld equation than the temporal formulation although the resulting algebraic eigenvalue problem may be more difficult to solve. This suggests that many of the solutions to the temporal formulation presented in the literature should be redeveloped for the correct spatial formulation.

It is unfortunate that the theorem of Squire, who proved that the two-dimensional mode was the first to become unstable in any plane parallel flow, can only be proved in general for temporally growing disturbances. Thus one is not certain whether spatially growing two-dimensional disturbances will ever be observed, as the flow may be more unstable with respect to three-dimensional disturbances. However, Gaster (1970) has recently proved for inviscid flows that the most highly amplified spatially growing disturbance corresponds to the two-dimensional mode. For particular viscous flows it is possible to prove that spatially growing three-dimensional disturbances grow less rapidly than two-dimensional disturbances; however, no general proof exists as yet. Of course the fact that Squire's theorem cannot be proved in general for spatially growing disturbances does not imply that two-dimensional disturbances are not the mode observed. Indeed for the case of falling film flow considered here, we have experimental verification that two-dimensional waves are observed. Thus for this flow, solutions for two-dimensional spatially growing disturbances certainly yield useful results.

An interesting consequence of the spatial growth formulation of the linear stability problem for falling films is that it admits a solution for $\omega_r = 0$, that is, for stationary waves. Although several investigators have observed stationary waves on thin liquid film flow, no one has attempted to predict their occurrence. This little known instability mode for thin liquid film flow has eluded theoreticians because of their preoccupation with the temporal growth formulation. The latter formulation cannot admit a stationary wave solution since the transformation given by Equation (5) is indeterminant for a wave having zero frequency or equivalently zero phase velocity. Stationary wave formation on thin liquid film flow has been described in the thesis of Owens (1972) and will be discussed in the forthcoming paper (Krantz and Owens, 1973).

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NOTATION

A = dimensionless wave amplitude, one-half peak-to-peak
c = dimensionless complex wave velocity, $c^*/\overline{U}$
c_i = imaginary part of complex wave velocity
c_r = real part of complex wave velocity
g = acceleration of gravity
h_0 = mean film thickness
i = $\sqrt{-1}$
N_{Re} = Reynolds number, $\overline{U}h_0/\nu$
N_{We} = Weber number, $\sigma/\rho h_0 \overline{U}^2$
N_ζ = surface tension group, $(3/g\nu^4)^{1/3}/\rho$
Q = volumetric flow rate per unit width
t = dimensionless time, $t^*\overline{U}/h_0$
U = dimensionless mean flow velocity, $U^*/\overline{U}$
$\overline{U}$ = average mean flow velocity, Q/h_0
x = dimensionless streamwise coordinate, x^*/h_0
y = dimensionless cross-stream coordinate, y^*/h_0

Greek Letters

α = dimensionless complex wave number, $2\pi h_0/\lambda_r$
α_i = imaginary part of complex wave number
α_r = real part of complex wave number
β = angle of plane to horizontal
η = dimensionless instantaneous wave amplitude, η^*/h_0
λ_r = wave length
μ = viscosity
ν = kinematic viscosity, μ/ρ
ρ = density
σ = surface tension
ϕ = amplitude of dimensionless stream function, $\phi^* h_0/\overline{U}$
ψ = dimensionless stream function, $\psi^* h_0/\overline{U}$
ω_r = dimensionless angular frequency, $\omega^* h_0/\overline{U}$

Superscript

$*$ = dimensional quantity

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